ordinary / singular
regular / irregular

6.3 - Solutions About Singular Points

A singular point for the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ (*) is a point $x = x_0$ such that $a_2(x_0) = 0$. For our work with solving differential equations, we will only consider $x_0 = 0$. Standard form for (*) is $y'' + P(x)y' + Q(x)y = 0$. Multiplying by x^2 (or by $(x - x_0)^2$ if $x_0 \neq 0$) yields
$x^2y'' + x^2P(x)y' + x^2Q(x)y = 0$. By relabeling $p(x) = xP(x)$, $q = x^2Q(x)$, we get $x^2y'' + xp(x)y' + q(x)y = 0$.

Definition: $x = 0$ is a **regular singular point** if $p(x)$ and $q(x)$ are analytic at $x = 0$ and an **irregular singular point** if either $p(x)$ or $q(x)$ is not analytic at $x = 0$.

Example: Determine the singular points of the given differential equation. Classify each singular point as regular or irregular.

$x(x + 3)^2y'' - y = 0$

$x=0, -3$ are singular

To classify them as regular or irregular, write standard form, then examine them individually

$$y'' - \frac{1}{x(x+3)^2}y = 0$$

$x=0$: Mult by x^2 $\quad x^2y'' - \frac{x}{(x+3)^2}y = 0$

$P(x)=0$, $Q(x)$ is analytic at $x=0$.

$x=-3$: mult by $(x+3)^2$ $\quad (x+3)^2y'' - \frac{1}{x}y = 0$

$P(x)=0$, $Q(x)$ is analytic at $x=-3$.

Both $x=0, -3$ are regular singular points.

Example: Determine the singular points of the given differential equation. Classify each singular point as regular or irregular.

$$y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$$

$x=0, 1$ are singular points

$x=0$ is regular

not analytic at $x=1$

$(x-1)^2$: $(x-1)^2 y'' - \frac{(x-1)^2}{x} + \frac{1}{x-1} y = 0$ $x=1$ is irregular

Theorem: (Frobenius' Theorem) If $x = x_0$ is a regular singular point of $(*)$, then there is at least one solution of the form
$y = (x-x_0)^r \sum_{n=0}^{\infty} c_n(x-x_0)^n = \sum_{n=0}^{\infty} c_n(x-x_0)^{n+r}$, where the number r is a constant to be determined.

For $x=0$, this is $\sum_{n=0}^{\infty} c_n x^{n+r}$

Example: $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity do not differ by an integer. Use the method of Frobenius to obtain two linearly independent solutions about $x = 0$. Form the general solution on $(0, \infty)$.

What if $r = \frac{1}{2}$?
$c_0 x^{1/2} + c_1 x^{3/2} + c_2 x^{5/2}$

$$2xy'' + 5y' + xy = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

the index does not change when differentiating using the method of Frobenius

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} 5(n+r) c_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

• Exponents x^{k+r-1}

$$\sum_{k=0}^{\infty} 2(k+r)(k+r-1)c_k x^{k+r-1} + \sum_{k=0}^{\infty} 5(k+r)c_k x^{k+r-1}$$

$$+ \sum_{k=2}^{\infty} c_{k-2}\, x^{k+r-1} = 0$$

$\Rightarrow 2r(r-1)+5r=0$

• indices

$k=0$ $\quad 2r(r-1)c_0 x^{r-1} + 5rc_0 x^{r-1} = 0$

$k=1$ $\quad 2(r+1)r\, c_1 x^{r} + 5(r+1)c_1 x^{r} = 0$

$$\sum_{k=2}^{\infty} \left\{\left[2\underline{(k+r)}(k+r-1)+5\underline{(k+r)}\right]c_k + c_{k-2}\right\} x^{k+r-1} = 0$$

$(k+r)(2k+2r-2+5)$

We use the $\underline{k=0}$ term to find r

$(2r^2-2r+5r)=0 \Rightarrow 2r^2+3r=0$

$r(2r+3)=0 \Rightarrow r=0, -\frac{3}{2}$

$k=1$: $\quad (2r^2+7r+5)c_1=0 \Rightarrow (2r+5)(r+1)\,c_1=0$

either $r=-\frac{5}{2}, -1$ or $\underline{c_1=0}$

$$c_k = -\frac{1}{(k+r)(2k+2r+3)}\, c_{k-2}, \quad k=2,3,4,\ldots$$

$\underline{r=0}$

$$c_k = -\frac{1}{k(2k+3)}\, c_{k-2}$$

$\underline{r=-\frac{3}{2}}$

$$c_k = -\frac{1}{(k-\frac{3}{2})2k}\, c_{k-2}$$

$$c_k = -\frac{1}{k(2k-3)}\, c_{k-2}$$

$k=2 \quad c_2 = -\frac{1}{14} c_0$

$k=3 \quad c_3 = (\quad) \cancel{c_1} \to 0$

$k=4 \quad c_4 = -\frac{1}{44} c_2 = \frac{1}{616} c_0$

$k=2 \quad c_2 = -\frac{1}{2} c_0$

$k=3 \quad c_3 = (\quad) \cancel{c_1} \to 0$

$k=4 \quad c_4 = -\frac{1}{20} c_2 = \frac{1}{40} c_0$

Let $C_1 = c_0$ Let $C_2 = c_0$

$$\boxed{y = C_1 \left(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 + \cdots\right) + C_2 x^{-3/2}\left(1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 + \cdots\right)}$$

x^0 Frobenius

The equation $r(2r+3)c_0 = 0$ is an indicial equation and can be generalized as follows:

Starting with $x^n y'' + a_1(x)y' + a_0(x)y = 0$, $n = 1$ or 2

Obtain standard form $y'' + P(x)y' + Q(x)y = 0$

Clear any potential factors of x in the denominator

$$x^2 y'' + x^2 P(x) y' + x^2 Q(x) y = 0 \qquad (13)$$

Make a cosmetic change: $x^2 y'' + x p(x) y' + q(x) y = 0$

using the substitutions $p(x) = xP(x)$, $q(x) = x^2 Q(x)$

Then for $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and appropriate y', y''

(13) becomes

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + (a_0+a_1x+a_2x^2+\cdots)\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}$$

$$+(b_0+b_1x+b_2x^2+\cdots)\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\Big[(n+r)(n+r-1)+(a_0+a_1x+a_2x^2+\cdots)(n+r)$$
$$+(b_0+b_1x+b_2x^2+\cdots)\Big]c_n x^{n+r} = 0$$

For $n=0$, $[r(r-1)+a_0 r+b_0]c_0$ is the coefficient of the term with the smallest exponent on x.

So $r(r-1)+a_0r+b_0=0$ ← indicial equation

where a_0 is the constant of $p(x)$ and b_0 is the constant of $q(x)$.

Applied to the previous problem, we have

$2xy''+5y'+xy=0$ becomes

$$x^2y''+\frac{5}{2}xy'+\frac{x^2}{2}y=0$$

$$x^2y''+x\left(\frac{5}{2}y'\right)+\frac{x^2}{2}y=0$$

$p(x) = \frac{5}{2} \Rightarrow a_0 = \frac{5}{2}$ $\quad$ $q(x) = \frac{x^2}{2} \Rightarrow b_0 = 0$

Indicial equation: $r(r-1) + \frac{5}{2}r + 0 = 0$

$\Rightarrow 2r(r-1) + 5r = 0$

$2r^2 + 3r = 0 \Rightarrow r = 0, -\frac{3}{2}$

Example: Use the method of Frobenius to obtain two linearly independent solutions about the regular singular point $x = 0$.

$9x^2y'' + 9x^2y' + 2y = 0$

Find indicial roots: $x^2y'' + \overbrace{x^2y'}^{x\,p(x)\,y'} + \frac{2}{9}y = 0$

$p(x) = x \Rightarrow a_0 = 0$ $\quad$ $q(x) = \frac{2}{9} = b_0$

indicial equation: $r(r-1) + 0r + \frac{2}{9} = 0$

$9r^2 - 9r + 2 = 0$

$(3r-1)(3r-2) = 0 \Rightarrow r = \frac{1}{3}, \frac{2}{3}$

Note that these do not differ by an integer.

Back to $9x^2y'' + 9x^2y' + 2y = 0$:

$y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $\quad y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$

[6.2: $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 \ldots \Rightarrow y' = c_1 + 2c_2x + 3c_3x^2 + \cdots = \sum_{n=1}^{\infty} nc_nx^{n-1}$]

6.3: $y = \sum_{n=0}^{\infty} c_n x^{n+r} = c_0x^r + c_1x^{1+r} + \cdots$

$$\sum_{n=0}^{\infty} 9(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} 9(n+r)c_n x^{n+r+1}$$

$$+ \sum_{n=0}^{\infty} 2c_n x^{n+r} = 0$$

Let $k=n+1$
$n=k-1$

• Exponents (x^{k+r})

$$\sum_{k=0}^{\infty} 9(k+r)(k+r-1)c_k x^{k+r} + \sum_{k=1}^{\infty} 9(k+r-1)c_{k-1} x^{k+r}$$

$$+ \sum_{k=0}^{\infty} 2c_k x^{k+r} = 0$$

• Indices

$k=0$: $[9r(r-1)+2]c_0 = 0 \Rightarrow 9r^2 - 9r + 2 = 0$

$\Rightarrow r = \frac{1}{3}, \frac{2}{3}$ (as before ☺)

For $k \geq 1$, $[9(k+r)(k+r-1)+2]c_k + 9(k+r-1)c_{k-1} = 0$

$$c_k = -\frac{9(k+r-1)}{9(k+r)(k+r-1)+2} c_{k-1}, \quad k = 1, 2, 3, \dots$$

$r = \frac{1}{3}$	$r = \frac{2}{3}$
$c_k = -\frac{9(k-\frac{2}{3})}{9(k+\frac{1}{3})(k-\frac{2}{3})+2} c_{k-1}$	$c_k = -\frac{9(k-\frac{1}{3})}{9(k+\frac{2}{3})(k-\frac{1}{3})+2} c_{k-1}$
$c_k = -\frac{3(3k-2)}{(3k+1)(3k-2)+2} c_{k-1}$	$c_k = -\frac{3(3k-1)}{(3k+2)(3k-1)+2} c_{k-1}$

$$c_k = -\frac{3(3k-2)}{9k^2-3k} c_{k-1}$$

$$c_k = -\frac{\cancel{3}(3k-2)}{\cancel{3}k(3k-1)} c_{k-1}$$

$$c_k = -\frac{3(3k-1)}{9k^2+3k} c_{k-1}$$

$3k(3k+1)$

$$c_k = -\frac{3k-1}{k(3k+1)} c_{k-1},$$

$k = 1, 2, 3, \ldots$

$(r = \frac{1}{3})$

$k=1: \; c_1 = -\frac{1}{2} c_0$

$k=2: \; c_2 = -\frac{4}{2(5)} c_1 = \frac{1}{5} c_0$

$C_1 = C_0$

$(r = \frac{2}{3})$

$k=1: \; c_1 = -\frac{2}{4} c_0 = -\frac{1}{2} c_0$

$k=2: \; c_2 = -\frac{5}{2(7)} c_1 = \frac{5}{28} c_0$

$C_2 = C_0$

$$y = C_1 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{5}x^2 + \cdots\right) + C_2 x^{2/3}\left(1 - \frac{1}{2}x + \frac{5}{28}x^2 + \cdots\right)$$

$$\boxed{y_1 = x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{5}x^2 + \cdots\right), \quad y_2 = x^{2/3}\left(1 - \frac{1}{2}x + \frac{5}{28}x^2 + \cdots\right)}$$

If r_1 and r_2 differ by an integer–that is, if $r_2 - r_1 \in \mathbb{Z}$–we can use r_1 to find y_1 and then reduction of order to find r_2.

Example: $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Use the method of Frobenius to obtain at least one series solution about $x = 0$. Use reduction of order and a CAS to find a second solution. Form the general solution on $(0, \infty)$.

$$y'' + \frac{3}{x}y' - 2y = 0$$

indicial roots: $x^2 y'' + x\,3y' - 2x^2 y = 0$

$p(x) = 3 \Rightarrow a_0 = 3 \qquad q(x) = -2x \Rightarrow b_0 = 0$

$r(r-1) + 3r = 0 \Rightarrow r^2 + 2r = 0$

$r = -2, 0$ (these differ by an integer)

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r-2}$$

$$- \sum_{n=0}^{\infty} 2c_n x^{n+r} = 0$$

• Exponents:

$$\sum_{k=0}^{\infty} \left[(k+r)(k+r-1) + 3(k+r)\right] c_k x^{k+r-2} - \sum_{k=2}^{\infty} 2c_{k-2} x^{k+r-2} = 0$$

$(k+r)(k+r+2)$

$k=0: \; r(r+2)c_0 = 0 \Rightarrow r = 0, -2$

$k=1: \; (\text{stuff w/r})\, c_1 = 0 \Rightarrow c_1 = 0$

$$(k+r)(k+r+2)c_k - 2c_{k-2} = 0, \quad k = 2,3,4,\ldots$$

$$c_k = \frac{2}{(k+r)(k+r+2)} c_{k-2}, \quad k = 2,3,4,\ldots$$

$r=-2$

$$c_k = \frac{2}{k(k-2)} c_{k-2}$$

$r=0$

$$c_k = \frac{2}{k(k+2)} c_{k-2}$$

$k=2 \quad c_2 = \frac{1}{4} c_0$

$k=3 \quad c_3 = \frac{2}{15} c_1 = 0$

$k=4 \quad c_4 = \frac{1}{12} c_2 = \frac{1}{48} c_0$

$$y_1 = 1 + \frac{1}{4}x^2 + \frac{1}{48}x^4 + \cdots$$

This is the "at least one" power series solution

Recall reduction of order:

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

Std form gives $P(x) = \frac{3}{x} \Rightarrow e^{-\int \frac{3}{x}dx} = x^{-3}$

$$y_2 = y_1 \int \frac{x^{-3}}{(1+\frac{1}{4}x^2+\frac{1}{48}x^3+\cdots)^2}\,dx$$

$$y_2 = y_1 \int \frac{1}{x^3(1+\frac{1}{4}x^2+\frac{1}{48}x^3+\cdots)^2}\,dx$$

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$$y_2 = y_1\left(-\frac{1}{2}\ln x - \frac{1}{2x^2} + \frac{7x^2}{96} + \cdots\right)$$

General solution: $y = c_1 y_1 + c_2 y_2$